Iterators and Interaction

Ian Mackie [1]

LIX, École Polytechnique
91128 Palaiseau Cedex, France

Jorge Sousa Pinto, Miguel Vilaça [2]

CCTC / Departamento de Informática
Universidade do Minho
4710-057 Braga, Portugal

Abstract

We propose a method for encoding iterators (and recursion operators in general) using interaction nets. There are two main applications for this: the method can be used to obtain a visual notation for functional programs, in a visual programming system; and it can be used to extend the existing translations of the λ-calculus into interaction nets (that have been proposed as efficient implementation mechanisms) to languages with recursive types. This work can also be seen as a study of the relation between interaction net programming and functional programming.

1 Introduction

Interaction nets have been extensively used to produce new, efficient implementation mechanisms for the λ-calculus [9,14,15]. On the other hand, the use of visual notations for functional programs has long been an active research topic, whose main goal is to have a notation that can be used (i) to define functional programs visually, and (ii) to animate visually the execution of functional programs.

In this paper we propose a graphical system for functional programming, based on token-passing interaction nets. The system offers an adequate solution for classic problems of visual notations, including the treatment of higher-order functions, pattern-matching, and recursion (based on the use of recursion operators). The system implements a call-by-name semantics, with a straightforward correspondence between functional programs and graphical objects.

Most approaches to visual programming simply propose a notation for programs. Program evaluation is animated by representing visually the intermediate programs

[1] Email:mackie@lix.polytechnique.fr

[2] Emails: {jsp,jmvilaca}@di.uminho.pt. The work of the 3rd. author was funded by FCT grant SFRH / BD / 18874 / 2004.

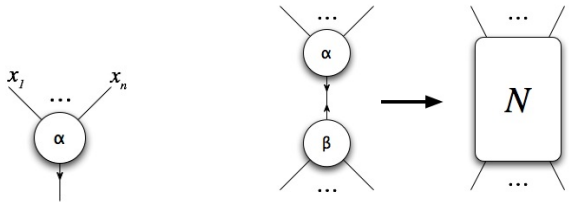


Fig. 1. An agent and an interaction rule

that result from executing reduction steps on the initial program, using the operational semantics of the underlying functional language. Our approach differs from this in that we use a graph-rewriting formalism with its own operational semantics.

Technically the main contribution of the paper is an extension of Sinot's token-passing implementation of the λ-calculus [19] to typed languages with recursive types and recursive function definitions based on recursion operators. We illustrate our ideas using the simply-typed λ-calculus with booleans, natural numbers, lists, and their respective iterators, but in fact the system can be extended smoothly to arbitrary polynomial types. Call-by-name evaluation is used for this language, but call-by-value and call-by-need could easily be obtained by building on previous results by Sinot. The token-passing encoding of the λ-calculus, to be combined with the encoding of recursion patterns proposed here, is just meant to be an illustrative choice for its simplicity and standard strategies.

An interesting feature of the work presented in this paper is that it results in interaction systems that are very similar to the typical examples of ("direct") interaction net programs. In this sense our work justifies semantically a functional subset of interaction nets. Moreover this provides further evidence that our approach is indeed an appropriate and natural way to represent functional programs visually.

The paper is structured as follows: Sections 2 and 3 contain background material on visual programming with interaction nets and on the token-passing encoding of the λ-calculus. Section 4 defines the functional language used in the paper. Sections 5 and 6 contain the translation of functional programs into token-passing nets and examples of its use. Section 7 considers extensions of the language with other recursion operators. We conclude the paper in Section 8.

2 Interaction Nets

Interaction nets [12] are constrained graph rewriting systems that can still encode all the computable functions. Interaction nets provide a model of computation in a graphical setting. Programs are represented as particular kinds of graphs, and computation is expressed as graph transformations. Interaction net systems are user-defined, in the same way as term rewriting systems, by giving a signature Σ (a set of symbols with a given arity) and a set of interaction rules R. An occurrence of a symbol is called an *agent*. An agent with arity n has $n+1$ *ports*: a distinguished one, depicted by an arrow, called the principal port, and n auxiliary ports. Agents are represented graphically as shown in Figure 1, on the left.

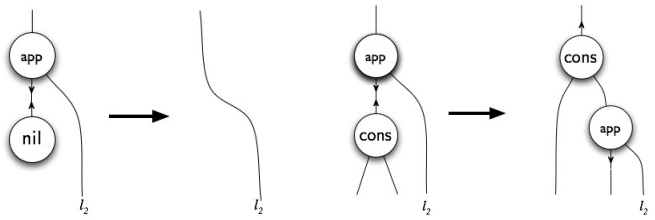
A net N built on a signature Σ is a graph (not necessarily connected) with agents at the vertices. The edges of the net connect agents together at the ports

such that there is only one edge at every port. Edges may connect two different ports of the same agent. The ports of an agent that are not connected to another agent are called the *free ports* of the net.

A pair of agents, say (α, β), connected on their principal ports is called an *active pair*, which is the interaction net analogue of a redex. An interaction rule replaces an occurrence of the active pair (α, β) by a net N. The rule has to satisfy a very strong condition: all the free ports are preserved during reduction, and moreover there is at most one rule for each pair of agents. The diagram shown on the right in Figure 1 illustrates the idea, where N is any net built from the signature.

An interaction net system is therefore fully defined by the pair (Σ, R). We say that a net is in normal form if it does not contain any active pairs. We use the notation $\Longrightarrow$ for one-step reduction and $\Longrightarrow^*$ for its transitive reflexive closure. Additionally, we write $N \Downarrow N'$ if there is a sequence of interaction steps $N \Longrightarrow^* N'$, such that N' is a net in normal form. The strong constraints on the definition of interaction rules imply that reduction commutes (the one-step diamond property holds), and thus confluence is easily obtained. Consequently, any normalizing interaction net is strongly normalizing.

The advantages of using interaction nets for visual programming can be understood by looking at a simple example. The following interaction rules define visually the behaviour of the list concatenation operation.



where the symbol app is used for concatenation agents, and nil and cons are the expected list constructors. The principal port of app is connected to the first list argument, and the result of the operation is obtained in the auxiliary port shown on top. This form of visual programming can be summarized as follows.

- Both programs and data are represented in the same simple graphical formalism.
- Programs can be animated without leaving the interaction formalism: instead of resorting to an external interpreter and then displaying the result of each evaluation step, a program can be animated by simply reducing the net. The reader can try this by connecting two lists of some type to an app agent and then applying the rules given above.
- Pattern-matching for external constructors is in-built.
- Recursive definitions are expressed very naturally as interaction rules involving agents (such as app) that are reintroduced on the right-hand side. Rule application then corresponds to the expansion of a recursive definition.

The above example is functional in nature: app can be written in a straightforward way as a function of two arguments that performs recursion on its first argument. But the interaction net formalism does not offer a satisfactory semantic interpretation for the behaviour of that symbol. Moreover, many interaction net

systems can be defined that do not have this functional reading.

What is missing is a clear correspondence between functional definitions and interaction systems like the one shown. In this paper we establish a correspondence between agents with "obviously functional" interaction rules like those given for app and functions defined with recursion operators.

We remark that the inherent inability of interaction nets to match constructors at a level deeper than one raises no problems: the simple form of pattern-matching available in interaction nets is sufficient to capture the behaviour of many powerful operators, such as recursors and accumulations, as will be shown in Section 7.

3 The Token-passing Encoding of the λ-calculus

A number of different translations of the λ-calculus into interaction nets exist. These have in common some basic principles:

- Terms are translated into nets of a fixed interaction net system.
- Variables are translated simply as edges in $\mathcal{T}(t)$.
- If t is a closed λ-term then the net $\mathcal{T}(t)$ has one free port, corresponding to the *root* of the term, which will be drawn at the top of the net. If not, and $x_1 \dots x_n$ are free variables in t, then the net $\mathcal{T}(t)$ has n additional free ports (represented at the bottom) corresponding to each of the variables.
- $\mathcal{T}(\lambda x.t)$ is a net constructed structurally from $\mathcal{T}(t)$. This introduces an abstraction symbol λ at the root of the term, with ports linked to the edge representing the bound variable x and to the root of the abstraction body net, $\mathcal{T}(t)$. The special case of $x \notin \mathcal{FV}(t)$ is handled by introducing an *erasing agent* ε.
- $\mathcal{T}(t\,u)$ is a net constructed structurally from $\mathcal{T}(t)$ and $\mathcal{T}(u)$. This introduces an application symbol @ with ports connected to the root ports of $\mathcal{T}(t)$ and $\mathcal{T}(u)$. The special case of a free variable occurring in both terms is handled by introducing a *copying agent* c, with its two auxiliary ports connected to the edges representing the free variable in $\mathcal{T}(t)$ and $\mathcal{T}(u)$, and the edge connected to its principal port represents the variable in $\mathcal{T}(t\,u)$.

The *token-passing* encodings [19] use an interaction system where two different symbols exist for application: one is the syntactic symbol @ introduced by the translation; the corresponding agents have their principal ports facing the root of the term and will be depicted by triangles. A second symbol $\widehat{@}$ exists that will be used for computation; to simplify the figures, the corresponding agents will be depicted by circles equally labelled with @. Their principal ports face the net that represents the applied function, to make possible interaction with λ agents.

The translation $\mathcal{T}_{\text{tp}}(\cdot)$ encodes terms in the system $(\Sigma_{\text{tp}}, R_{\text{tp}})$ where $\Sigma_{\text{tp}} = \{\Downarrow, @, \widehat{@}, \lambda, c, \varepsilon, \delta\}$. The translation is shown in Figure 2 where $\mathcal{T}(.)$ stands for $\mathcal{T}_{\text{tp}}(\cdot)$. It generates nets containing no active pairs, so no reduction can happen. The special symbol $\Downarrow$ is used as an evaluation *token*: an agent $\Downarrow$ traverses the net, transforming occurrences of @ into $\widehat{@}$, thus triggering reductions. The evaluation rules involving $\Downarrow$ can be tailored to a specific evaluation strategy. For call-by-name, R_{tp} consists of the rules in Figure 3 (the arity of each symbol can be inferred from the rules). This

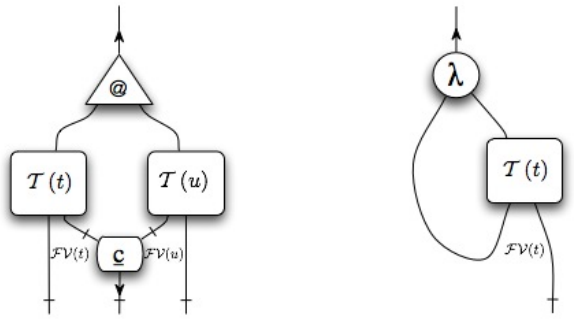


Fig. 2. The token-passing translation of λ-terms: the nets $\mathcal{T}(t\,u)$ and $\mathcal{T}(\lambda x.t)$. $\underline{\mathrm{c}}$ denotes an array of c agents, one for each free variable occurring in both t and u. In $\mathcal{T}(\lambda x.t)$, a special case exists (not depicted) when the bound variable does not occur in the term: an ε agent must be connected to the λ agent instead.

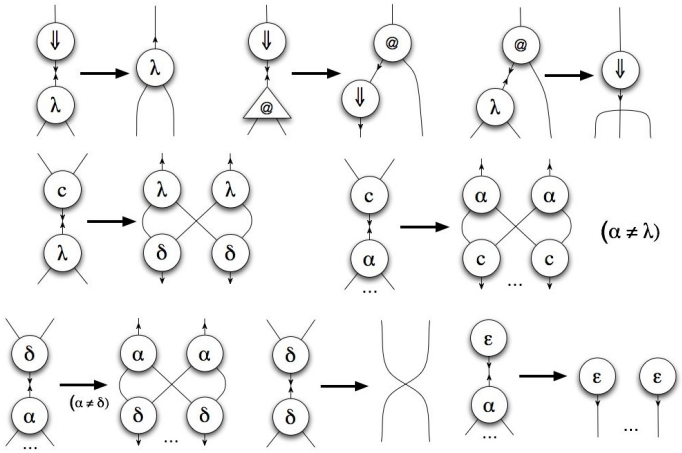


Fig. 3. The token-passing rules R_{tp}. Note the rule *templates* for (c, α), (δ, α), and (ε, α), which generate different rules for each instance of the agent α.

comprises evaluation rules involving ⇓, a computation rule involving @ and λ, and management (copying and erasing) rules. The symbol δ is a mutation of c used for copying abstractions.

To start the reduction (corresponding to normal order evaluation), a ⇓ symbol must be connected to the root port of the term. Let $\Downarrow N$ denote the net obtained by connecting a ⇓ agent to the root port of N; then the following correctness result holds: $t \Downarrow z$ iff $\Downarrow \mathcal{T}_{\mathrm{tp}}(t) \longrightarrow^* \mathcal{T}_{\mathrm{tp}}(z)$, where the evaluation relation $\cdot \Downarrow \cdot$ is defined by the standard evaluation rules for call-by-name:

$$\frac{}{\lambda x.t \Downarrow \lambda x.t} \qquad \frac{t \Downarrow \lambda x.t' \qquad t'[u/x] \Downarrow z}{t\,u \Downarrow z}$$

4 The language BNL

The language used in this paper is the simply-typed λ-calculus extended with natural numbers, booleans, lists, and iterators for these recursive types. BNL is defined by the following syntax for types and terms (x, y range over a set of variables):

$$\begin{array}{rcl} \tau, \sigma & ::= & \mathbf{Bool} \mid \mathbf{Nat} \mid \mathbf{List}(\tau) \mid \tau \to \sigma \\ t, u, v & ::= & x \mid \lambda x.t \mid t\,u \mid \mathsf{tt} \mid \mathsf{ff} \mid \mathsf{iterbool}(t,u,v) \mid 0 \mid \mathsf{suc}(t) \mid \mathsf{iternat}(\lambda x.t, u, v) \\ & \mid & \mathsf{nil} \mid \mathsf{cons}(t,u) \mid \mathsf{iterlist}(\lambda xy.t, u, v) \end{array}$$

and by the typing rules given by:

$$\frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash \lambda x.t : \sigma \to \tau} \qquad \frac{\Gamma \vdash t : \sigma \to \tau \qquad \Gamma \vdash u : \sigma}{\Gamma \vdash t\,u : \tau}$$

$$\frac{}{\Gamma \vdash \mathsf{tt} : \mathbf{Bool}} \quad \frac{}{\Gamma \vdash \mathsf{ff} : \mathbf{Bool}}$$

$$\frac{}{\Gamma \vdash 0 : \mathbf{Nat}} \quad \frac{\Gamma \vdash t : \mathbf{Nat}}{\Gamma \vdash \mathsf{suc}(t) : \mathbf{Nat}} \quad \frac{}{\Gamma \vdash \mathsf{nil} : \mathbf{List}(\tau)}$$

$$\frac{\Gamma \vdash h : \tau \qquad \Gamma \vdash t : \mathbf{List}(\tau)}{\Gamma \vdash \mathsf{cons}(h,t) : \mathbf{List}(\tau)}$$

$$\frac{\Gamma \vdash t : \mathbf{Bool} \qquad \Gamma \vdash V : \tau \qquad \Gamma \vdash F : \tau}{\Gamma \vdash \mathsf{iterbool}(V, F, t) : \tau}$$

$$\frac{\Gamma \vdash t : \mathbf{Nat} \qquad \Gamma \vdash \lambda x.S : \tau \to \tau \qquad \Gamma \vdash Z : \tau}{\Gamma \vdash \mathsf{iternat}(\lambda x.S, Z, t) : \tau}$$

$$\frac{\Gamma \vdash t : \mathbf{List}(\sigma) \quad \Gamma \vdash \lambda xy.C : \sigma \to \tau \to \tau \quad \Gamma \vdash N : \tau}{\Gamma \vdash \mathsf{iterlist}(\lambda xy.C, N, t) : \tau}$$

The call-by-name evaluation semantics is as follows. Note that constructor terms of a given type are taken to be canonical forms.

$$\frac{}{\lambda x.t \Downarrow \lambda x.t} \qquad \frac{t \Downarrow \lambda x.t' \qquad t'[u/x] \Downarrow z}{t\,u \Downarrow z}$$

$$\frac{}{0 \Downarrow 0} \quad \frac{}{\mathsf{suc}(n) \Downarrow \mathsf{suc}(n)} \quad \frac{}{\mathsf{tt} \Downarrow \mathsf{tt}} \quad \frac{}{\mathsf{ff} \Downarrow \mathsf{ff}}$$

$$\frac{t \Downarrow \mathsf{tt} \qquad V \Downarrow z}{\mathsf{iterbool}(V, F, t) \Downarrow z} \qquad \frac{t \Downarrow \mathsf{ff} \qquad F \Downarrow z}{\mathsf{iterbool}(V, F, t) \Downarrow z}$$

$$\frac{t \Downarrow 0 \qquad Z \Downarrow z}{\mathsf{iternat}(\lambda x.S, Z, t) \Downarrow z}$$

$$\frac{t \Downarrow \mathsf{suc}(n) \qquad S[\mathsf{iternat}(\lambda x.S, Z, n)/x] \Downarrow z}{\mathsf{iternat}(\lambda x.S, Z, t) \Downarrow z}$$

$$\frac{}{\mathsf{nil} \Downarrow \mathsf{nil}} \qquad \frac{}{\mathsf{cons}(u,v) \Downarrow \mathsf{cons}(u,v)}$$

$$\frac{t \Downarrow \mathsf{nil} \qquad N \Downarrow z}{\mathsf{iterlist}(\lambda xy.C, N, t) \Downarrow z}$$

$$\frac{t \Downarrow \mathsf{cons}(u,v) \qquad C[u/x, \mathsf{iterlist}(\lambda xy.C, N, v)/y] \Downarrow z}{\mathsf{iterlist}(\lambda xy.C, N, t) \Downarrow z}$$

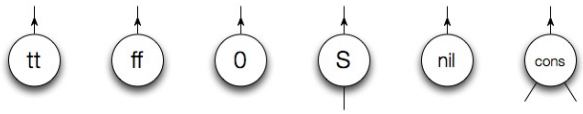
Some variables have been capitalized due to reasons that will become clear later on.

5 A Token-passing Encoding of BNL

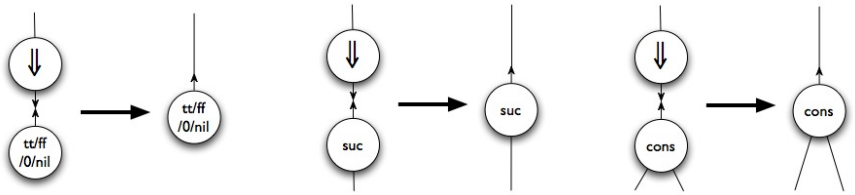
We extend to BNL the token-passing call-by-name translation of the λ-calculus into the interaction system $(\Sigma_{\mathrm{tp}}, R_{\mathrm{tp}})$. We first extend the interaction system and then the translation function. The novelty of this encoding is not the token-passing aspect (which is a natural extension of the encoding of the λ-calculus), but rather the approach to recursion.

We first consider data structures. Terms of inductively defined types can be represented in interaction nets in the natural way, as *trees* where each node corresponds to a constructor, with its principal port facing the parent node. In a (call-by-name) token-passing implementation, there will be an interaction rule between the token agent and each such constructor symbol that will stop evaluation—this corresponds to the fact that constructor terms are canonical forms.

For BNL we define the system $(\Sigma_{\mathrm{BNL}}, R_{\mathrm{BNL}})$ where Σ_{BNL} consists of the symbols tt, ff, 0 and nil with arity 0; suc with arity 1; and cons with arity 2, depicted as



and R_{BNL} consists of the rules given below.



Each recursive program will be encoded in an interaction system specifically generated for it. This is a major novelty of our approach. The interaction system for the λ-calculus will not be extended by introducing a fixed set of symbols; instead a new symbol will be introduced for *each occurrence of a recursion operator*, with an interaction rule for each different constructor of its argument type, so a dedicated interaction system (Σ_t^0, R_t^0) is generated for each term t.

This system is constructed by a recursive function $(\Sigma_t^0, R_t^0) = \mathcal{S}(t)$, defined as follows ($\cup$ is occasionally used to denote pairwise union).

$\mathcal{S}(x) \doteq \mathcal{S}(\mathsf{tt}) \doteq \mathcal{S}(\mathsf{ff}) \doteq \mathcal{S}(0) \doteq \mathcal{S}(\mathsf{nil}) \doteq (\emptyset, \emptyset)$

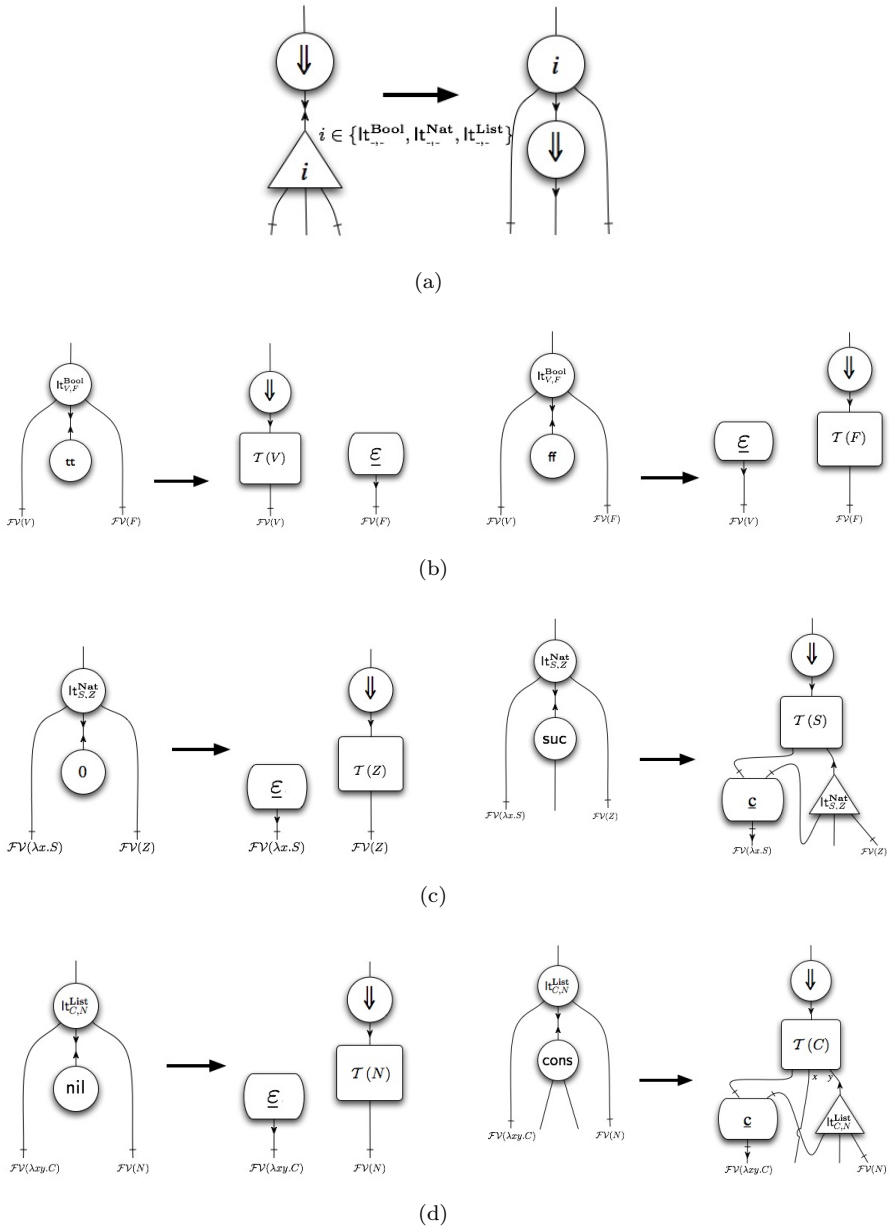
$\mathcal{S}(\lambda x.t) \doteq \mathcal{S}(\mathsf{suc}(t)) \doteq \mathcal{S}(t)$

$\mathcal{S}(t\,u) \doteq \mathcal{S}(\mathsf{cons}(t,u)) \doteq \mathcal{S}(t) \cup \mathcal{S}(u)$

$\mathcal{S}(\mathsf{iterbool}(V,F,b)) \doteq (\{\mathsf{lt}^{\mathbf{Bool}}_{V,F}, \widehat{\mathsf{lt}^{\mathbf{Bool}}_{V,F}}\} \cup \Sigma, R_{\mathsf{lt}^{\mathbf{Bool}}_{V,F}} \cup R)$,
where $(\Sigma, R) = \mathcal{S}(b) \cup \mathcal{S}(V) \cup \mathcal{S}(F)$, and $R_{\mathsf{lt}^{\mathbf{Bool}}_{V,F}}$ consists of the interaction rules included in Figures 4(a) and 4(b).

$\mathcal{S}(\mathsf{iternat}(\lambda x.S, Z, n)) \doteq (\{\mathsf{lt}^{\mathbf{Nat}}_{S,Z}, \widehat{\mathsf{lt}^{\mathbf{Nat}}_{S,Z}}\} \cup \Sigma, R_{\mathsf{lt}^{\mathbf{Nat}}_{S,Z}} \cup R)$
where $(\Sigma, R) = \mathcal{S}(n) \cup \mathcal{S}(S) \cup \mathcal{S}(Z)$ and $R_{\mathsf{lt}^{\mathbf{Nat}}_{S,Z}}$ consists of the interaction rules included in Figures 4(a) and 4(c).

$\mathcal{S}(\mathsf{iterlist}(\lambda xy.C, N, l)) \doteq (\{\mathsf{lt}^{\mathbf{List}}_{C,N}, \widehat{\mathsf{lt}^{\mathbf{List}}_{C,N}}\} \cup \Sigma, R_{\mathsf{lt}^{\mathbf{List}}_{C,N}} \cup R)$
where $(\Sigma, R) = \mathcal{S}(l) \cup \mathcal{S}(C) \cup \mathcal{S}(N)$ and $R_{\mathsf{lt}^{\mathbf{List}}_{C,N}}$ consists of the interaction rules included in Figures 4(a) and 4(d).



(a)

(b)

(c)

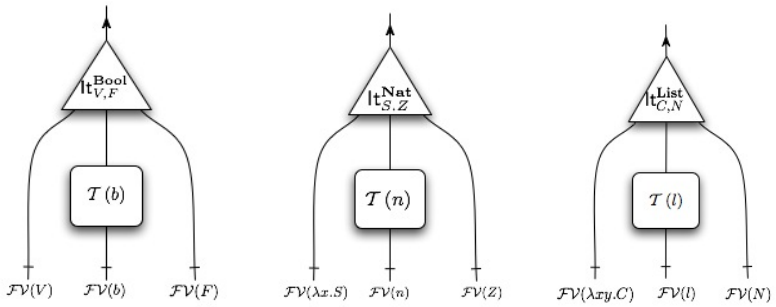
(d)

Fig. 4. Interaction rules for iterators

Iterator symbols are introduced in pairs $(\mathsf{It}^{\cdots}_{\cdots}, \widehat{\mathsf{It}}^{\cdots}_{\cdots})$ where the first symbol is used for syntactic agents and the second for computation agents (similarly to $@, \widehat{@}$). To simplify the graphical presentation, syntactic agents are depicted by triangles. The arity of each symbol can be inferred from the interaction rules. In Figures 4(b) to 4(d), $\underline{c}$ denotes an array of c agents and $\underline{\varepsilon}$ denotes an array of ε agents. The size of this arrays depends, respectively, on the number of shared and free variables in the corresponding terms.

A BNL program t will be translated into a net defined in the system $(\Sigma_t, R_t) = (\Sigma_{\text{tp}} \cup \Sigma_{\text{BNL}} \cup \Sigma_t^0, R_{\text{tp}} \cup R_{\text{BNL}} \cup R_t^0)$ where $(\Sigma_{\text{tp}}, R_{\text{tp}})$ was defined in Section 3.

Definition 5.1 Given a BNL program t, the net $\mathcal{T}(t)$ is given as follows.



(a) (b) (c)

Fig. 5. Translations of iterators. We remark that, if the same variable occurs in more than one of the named sets (say, $\mathcal{FV}(V)$ and $\mathcal{FV}(F)$ for $\mathsf{iterbool}(V, F, b)$), c agents must be used to group the edges, analogously to what happens in the encoding of an application $t\,u$ (see Figure 2).

- If t is an abstraction, variable or application, then $\mathcal{T}(t)$ is defined as in Section 3.
- If t is one of tt, ff, 0, or nil, then $\mathcal{T}(t)$ is an instance of the corresponding symbol.
- If $t = \mathsf{suc}(t')$, then $\mathcal{T}(t)$ is constructed by connecting the auxiliary port of a suc agent to the root port of $\mathcal{T}(t')$.
- If $t = \mathsf{cons}(h, t')$, then $\mathcal{T}(t)$ is constructed by connecting the auxiliary ports of a cons agent to the root ports of $\mathcal{T}(h)$ and $\mathcal{T}(t')$.
- If $t = \mathsf{iterbool}(V, F, b)$ then $\mathcal{T}(t)$ is given by the net in Figure 5(a).
- If $t = \mathsf{iternat}(\lambda x.S, Z, n)$ then $\mathcal{T}(t)$ is given by the net in Figure 5(b).
- If $t = \mathsf{iterlist}(\lambda xy.C, N, l)$ then $\mathcal{T}(t)$ is given by the net in Figure 5(c).

As is characteristic of token-passing implementations, all terms (including iterators) are translated as syntax trees. Syntactic iterator agents i are turned into their computation counterparts $\widehat{i}$ by token agents, in the same way as the @ agents in the encoding of the λ-calculus.

A first key aspect of our approach is that the interaction rules of the (computation) iterator agents internalise the iterator's parameters. For instance the net $\mathcal{T}(\mathsf{iterlist}(\lambda xy.C, N, \mathsf{cons}(h, t)))$ reduces to $\mathcal{T}(C[h/x, \mathsf{iterlist}(\lambda xy.C, N, t)/y])$, with an evaluation token on top to control call-by-name evaluation.

A second key aspect is that each such new symbol will have auxiliary ports in a one-to-one correspondence with the free variables in the iterator term, since iterator terms are not restricted to be closed. The significance of this will become clear from the examples. We end the section with a correctness result. The proofs can be found in a long version of this paper [1].

Lemma 5.2 *Let t be a closed* BNL *term; then: $t \Downarrow z \implies \Downarrow \mathcal{T}(t) \longrightarrow^* \mathcal{T}(z)$.*

Lemma 5.3 *Let t be a closed* BNL *term and z a canonical form, then: $\Downarrow \mathcal{T}(t) \longrightarrow^* \mathcal{T}(z) \implies t \Downarrow z$.*

Proposition 5.4 *(**Correctness**) If t is a closed* BNL *term and z a canonical form, then: $t \Downarrow z \iff \Downarrow \mathcal{T}(t) \longrightarrow^* \mathcal{T}(z)$.*

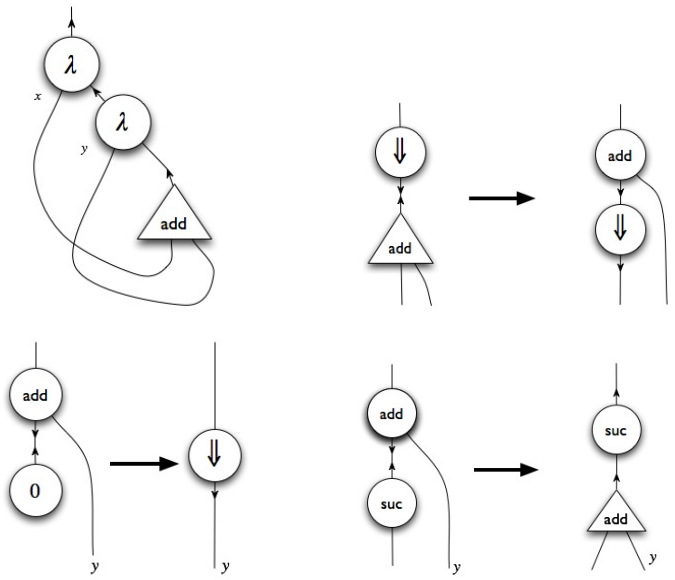


Fig. 6. Encoding of add and corresponding interaction rules

6 Examples

The following examples illustrate the use of the translation with different programs.

Example 6.1 Let $add = \lambda xy.\mathsf{iternat}(\lambda r.\mathsf{suc}(r), y, x)$ of type $\mathbf{Nat} \to \mathbf{Nat} \to \mathbf{Nat}$. The free variable y in the second argument of the iterator creates an auxiliary port in the symbol $\mathsf{It}^{\mathbf{Nat}}_{\mathsf{suc}(r),y}$. The net corresponding to the encoding of the function and the interaction rules generated are given in Figure 6, where add stands for $\mathsf{It}^{\mathbf{Nat}}_{\mathsf{suc}(r),y}$. We remark that the last rule, whose right-hand side contained an active pair, was normalized by reducing that pair. The same will happen in the following examples.

The interaction rules for the computation agent add constitute a highly intuitive visual definition of addition, as should happen in any framework for visual programming. An example evaluation of a program can be found in the appendix.

Example 6.2 The reader is invited to work out the encoding of the append function $app = \lambda l_1 l_2.\mathsf{iterlist}(\lambda hr.\mathsf{cons}(h, r), l_2, l_1)$ with type $\mathbf{List}(\tau) \to \mathbf{List}(\tau) \to \mathbf{List}(\tau)$, and to compare it to the rules given in Section 2 for the agent app as an example of a direct interaction net program.

Example 6.3 Our final example corresponds to a higher-order function, $map = \lambda fl.\mathsf{iterlist}(\lambda hr.\mathsf{cons}(f\ h, r), \mathsf{nil}, l)$ with type $\mathsf{map} : (\tau \to \sigma) \to \mathbf{List}(\tau) \to \mathbf{List}(\sigma)$. This example differs from the previous in that a free variable (f) now occurs in the *first* argument of the iterator. Again this generates an auxiliary port in $\mathsf{It}^{\mathbf{List}}_{\mathsf{cons}(f\,h,r),\mathsf{nil}}$. The function is encoded as the net in Figure 7, where the name map is used for the symbol $\mathsf{It}^{\mathbf{List}}_{\mathsf{cons}(f\,h,r),\mathsf{nil}}$. Its interaction rules are also shown in the figure.

Again the visual representation is intuitive. The role of the copying agent in the second rule is to produce two copies of the encoding of the function: one to be applied to the head of the list, and another to be used in the recursive call.

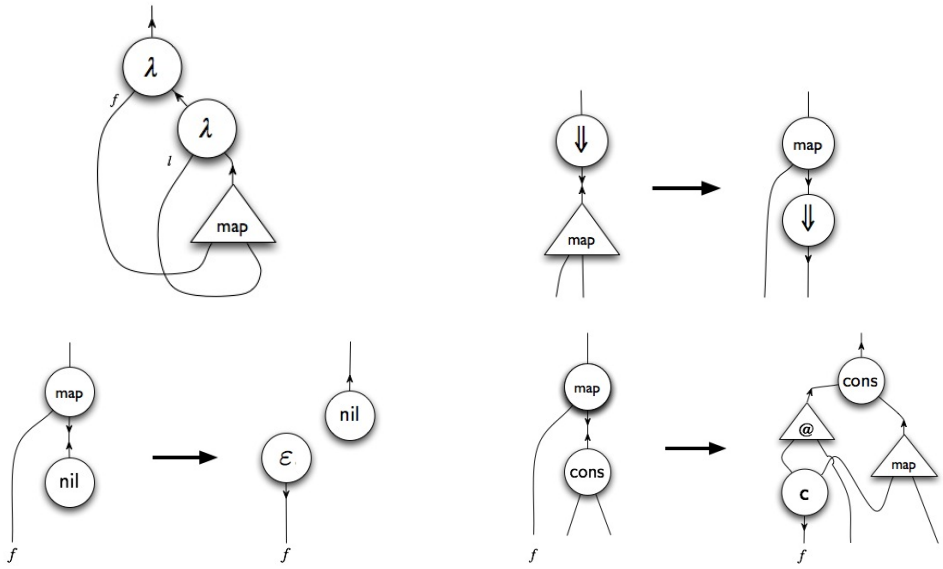


Fig. 7. Encoding of map and corresponding interaction rules

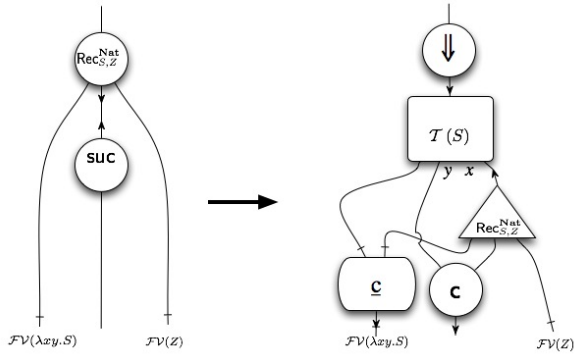


Fig. 8. Interaction rules for natural numbers recursor

7 Other Recursion Operators

A recursor for natural numbers can be added to the language with the following syntax, typing and evaluation rules: $t, u, v ::= \ldots \mid \mathsf{recnat}(\lambda xy.u, v, t)$,

$$\frac{\Gamma \vdash t : \mathbf{Nat} \qquad \Gamma \vdash \lambda xy.S : \tau \to \mathbf{Nat} \to \tau \qquad \Gamma \vdash Z : \tau}{\Gamma \vdash \mathsf{recnat}(\lambda xy.S, Z, t) : \tau}$$

$$\frac{t \Downarrow 0 \qquad Z \Downarrow z}{\mathsf{recnat}(\lambda xy.S, Z, t) \Downarrow z}$$

$$\frac{t \Downarrow \mathsf{suc}(n) \qquad S[\mathsf{recnat}(\lambda xy.S, Z, n)/x, n/y] \Downarrow z}{\mathsf{recnat}(\lambda xy.S, Z, t) \Downarrow z}$$

The computational power of this recursor operator comes from the fact that it has access to its *argument*, in addition to the recursive result on that argument. The factorial function, for instance, can be defined in this way, but not with an iterator. Replacing the iterator with this recursor requires only minor changes in

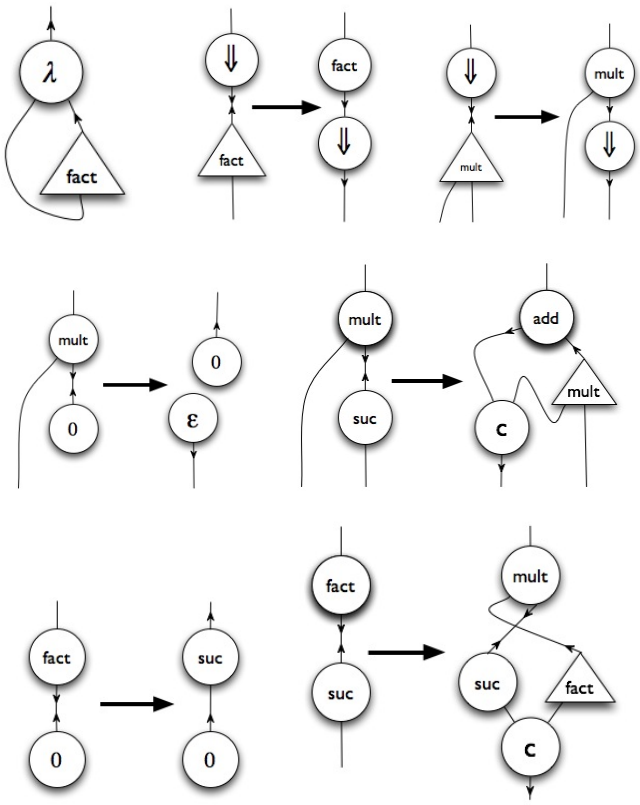


Fig. 9. Encoding of fact and corresponding interaction rules. Also includes rules for mult.

the interaction system: an agent $\mathsf{Rec}^{\mathbf{Nat}}_{S,Z}$ must be used in the translation of the expression $\mathsf{recnat}(\lambda xy.S, Z, t)$ instead of $\mathsf{It}^{\mathbf{Nat}}_{S,Z}$. Its interaction with the successor symbol is given by the rule shown in Figure8, where we note that for an argument $\mathsf{suc}(n)$, the net representing n must now be duplicated.

For instance, the translation of $\mathit{fact} = \lambda n.\mathsf{recnat}(\lambda xy.\mathit{mult}\,\mathsf{suc}(y)\,x, \mathsf{suc}(0), n)$ with multiplication defined as $\mathit{mult} = \lambda xy.\mathsf{iternat}(\lambda r.\mathit{add}\,y\,r, 0, x)$, is given in Figure 9, where the symbols *fact* and *mult* stand respectively for $\mathsf{Rec}^{\mathbf{Nat}}_{\mathit{mult}\,\mathsf{suc}(y)\,x,\mathsf{suc}(0)}$ and $\mathsf{It}^{\mathbf{Nat}}_{\mathit{add}\,y\,r,0}$. Notice the RHS of the rules are fully or partially reduced (optimized).

In a language where recursion is only available through the use of recursion operators, it is important to have a number of different such operators, each of which may be more convenient for writing certain families of programs. We take as example the Haskell `foldl` (left folding) list operator, which stores intermediate results in an accumulator argument, returned at the end of the list. Even though every program written with it can also be written with the more common `foldr` (right folding operator), it is still convenient to have it in the language. For instance, a linear time, tail-recursive function for reversing lists can be written in the two following ways:

```
revt l = foldr (\h r a -> r(h:a)) id l []

revt l = foldl (\r h -> h:r) [] l
```

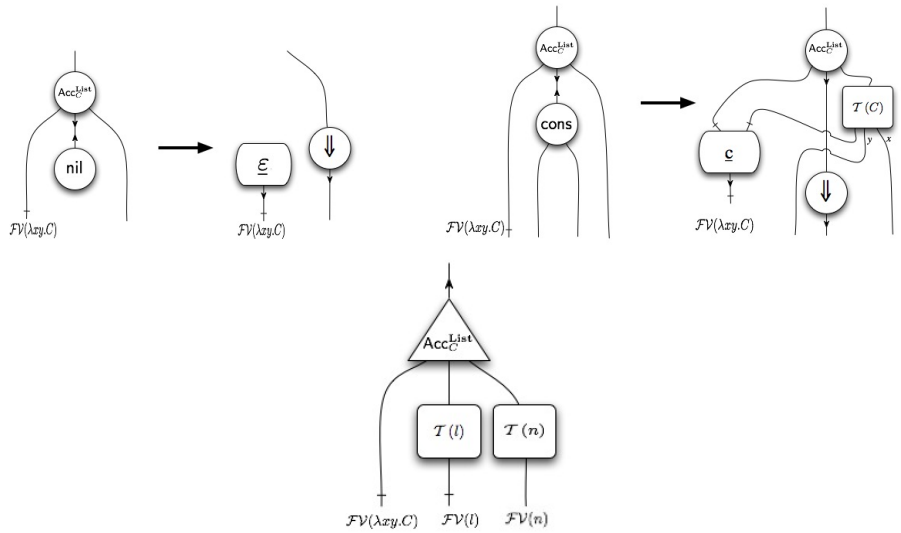


Fig. 10. Interaction rules for accumulations

The first version can be written in BNL. Applying the encoding of Section 5 results in a new agent $\mathsf{It}^{\mathbf{List}}_{(\lambda a.y\,\mathsf{cons}(x,a)),(\lambda x.x)}$. Naturally, the interaction rules for this agent introduce encodings of abstractions in their right-hand sides, which results in a quite complicated definition. To accommodate the second, clearly simpler definition, we now consider the extension of BNL with an *accumulation* operator similar to `foldl`, with the following typing and evaluation rules.

$$t, u, v ::= \ldots \mid \mathsf{acclist}(\lambda xy.t, u, v)$$

$$\frac{\Gamma \vdash t : \mathbf{List}(\tau) \qquad \Gamma \vdash \lambda xy.C : \sigma \to \tau \to \sigma \qquad \Gamma \vdash n : \sigma}{\Gamma \vdash \mathsf{acclist}(\lambda xy.C, n, t) : \sigma}$$

$$\frac{t \Downarrow \mathsf{nil} \qquad n \Downarrow z}{\mathsf{acclist}(\lambda xy.C, n, t) \Downarrow z}$$

$$\frac{t \Downarrow \mathsf{cons}(h, u) \qquad \mathsf{acclist}(\lambda xy.C, C[n/x, h/y], u) \Downarrow z}{\mathsf{acclist}(\lambda xy.C, n, t) \Downarrow z}$$

The function $\mathcal{S}(\cdot)$ that constructs the interaction system is extended as follows.

$$\mathcal{S}(\mathsf{acclist}(\lambda xy.C, n, l)) = (\{\mathsf{Acc}^{\mathbf{List}}_C, \widehat{\mathsf{Acc}^{\mathbf{List}}_C}\} \cup \Sigma, R_{\mathsf{Acc}^{\mathbf{List}}_C} \cup R)$$
$$\text{where } (\Sigma, R) = \mathcal{S}(l) \cup \mathcal{S}(C) \cup \mathcal{S}(n)$$

where $R_{\mathsf{Acc}^{\mathbf{List}}_C}$ consists of the rules of Figure 10, top (together with the obvious evaluation token rule). $\widehat{\mathcal{T}}(\mathsf{acclist}(\lambda xy.C, n, l))$ is then defined as the net shown in Figure 10, bottom. We remark that in the reduction rules for $\mathsf{acclist}(\lambda xy.C, n, l)$ the second argument n is not fixed throughout iteration; as such it cannot be internalized as part of the definition of the agent $\mathsf{Acc}^{\mathbf{List}}_C$. Instead the corresponding net is connected to an auxiliary port in that agent.

The list reversion function can now be written $\mathsf{revt} = \lambda l.\mathsf{acclist}(\lambda xy.\mathsf{cons}(y, x), \mathsf{nil}, l)$. The net $\widehat{\mathcal{T}}(\mathsf{revt})$ and the rules required are shown in Figure 11. Note that the symbol revt is used instead of $\mathsf{Acc}^{\mathbf{List}}_{\mathsf{cons}(y,x)}$.

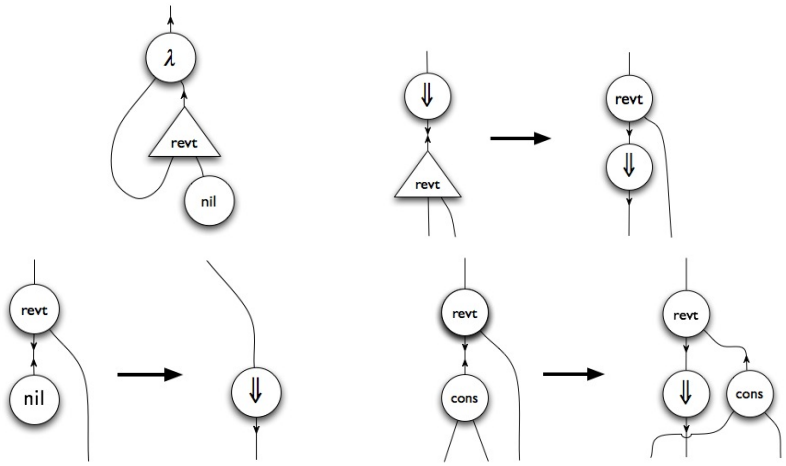


Fig. 11. Net and rules for list reversion

8 Conclusions and Future Work

We have presented an approach to encoding in interaction nets functional programs defined with recursion operators, and given the full details of the application of this approach to the token-passing implementation of a call-by-name language, which results in a very convenient visual notation for this language. The approach can be easily extended to richer sets of recursive types and other recursion operators, and also to new strategies. The novel characteristics of the encoding are (i) the fact that the interaction system is generated dynamically from the program, and (ii) the internalisation of some of the parameters of the recursion operator in the interaction rules of the symbol that encodes the operator's behaviour.

We have left types mostly out of our discussion. A net can be typed by assigning a type to every port. In our context, the types are those defined for the functional language BNL, except that they may occur either positively (in ports corresponding to data structures) or negatively (in ports corresponding to function or constructor arguments). In a correctly-typed net every edge connects two ports typed with $+A$ and $-A$ for some type A. So typing extends smoothly to the visual setting.

A prototype system for visual functional programming has been developed, integrated in the tool INblobs [3,20] for interaction net programming. The tool consists of an evaluator for interaction nets together with a visual editor and a compiler module that translates programs into nets. The latter module allows users to type in a functional program, visualize it, and then follow its evaluation visually step by step. The current compiler module is restricted to the iterators for $\mathbf{Bool}, \mathbf{Nat}$ and $\mathbf{List}(\tau)$, and automatically generates call-by-name (presented in this paper) or call-by-value systems. Additionally, a visual editing mode is available that allows users to construct nets corresponding to functional programs.

A topic that has been left out of the discussion in the paper is to give a direct (i.e. not resulting from a translation) characterization of the class of nets corresponding to recursive programs. This characterization could be used by the tool to restrict nets constructed visually to such a subclass of interaction nets. Also, the current implementation does not automatically normalize the RHS of the generated rules,

and moreover there is no way to convert visual programs back to textual ones.

A different line of work is inspired by work of the datatype-generic programming community and the school of program calculation [5]. This prompts the investigation of visual fusion laws for instance. Fusion laws simplify compositional functional programs before their application to arguments: before calculating $f(g(x))$ one may in certain conditions, by eliminating intermediate data structures, obtain a more efficient function h equivalent to $f \cdot g$, and calculate instead $h(x)$. A classic case is when g is an iterator. We conjecture that these laws can be proved in the interaction net setting by using notions of contextual equivalence [7]. Extending the visual programming tool with fusion capabilities would make possible to perform program transformations at the visual level. In [16] we investigated some preliminary ideas in this direction.

The token-passing translation of the λ-calculus has the advantage of implementing a simple evaluation order and maintaining a structure in the nets that is always immediately recognizable and understandable in terms of the evaluation semantics. As such it is totally appropriate for our goal of providing a visual representation for functional programs. Interaction nets have however also been extensively studied as an implementation mechanism for the λ-calculus. The main motivation for this approach is that it results in highly efficient evaluation strategies, made possible by the close control kept on the erasing and duplication of terms. The token-passing translation is not representative of most work in this area, which has concentrated on designing *efficient* translations. These translations are not controlled by an evaluation token (they produce nets already containing active pairs) and impose reduction strategies that cannot be defined using term-based abstract machines.

There are a number of interaction net encodings of the λ-calculus, which follow different strategies. To give just a sample, Gonthier, Abadi and Lévy [9] presented an implementation of optimal β-reduction. Mackie [14,15] has proposed several systems, each corresponding to a different strategy for reduction in the λ-calculus.

Let $\mathcal{T}(\cdot)$ be one such translation. Typically $\mathcal{T}(t\,u)$ is constructed from $\mathcal{T}(t)$ and $\mathcal{T}(u)$ by introducing an application symbol @ with its principal port connected to the root port of $\mathcal{T}(t)$. Our treatment of iterators can be adapted to this setting by removing the evaluator tokens and introducing the iterator agents with the principal port immediately facing the argument. When the iterated function is a closed term, a correctness result can be easily established: Let $\lambda x.S$ be a closed term, then

(i) $\mathcal{T}(\mathsf{iternat}(\lambda x.S, Z, 0)) \longrightarrow \mathcal{T}(Z)$

(ii) $\mathcal{T}(\mathsf{iternat}(\lambda x.S, Z, \mathsf{suc}(n))) \longrightarrow \mathcal{T}(S[\mathsf{iternat}(\lambda x.S, Z, n)/x])$

We remark that it is always possible to work with iterators with closed functions– thus this result applies to all programs.

For general terms, a correctness result has to be established for each translation, and it still has to be studied if, and in what way, the reduction strategy imposed by the translation for the λ-calculus is modified by this treatment of recursion.

References

[1] J. B. Almeida, I. Mackie, J. S. Pinto, and M. Vilaça. Encoding iterators in interaction nets. Available from `http://www.di.uminho.pt/~jmvilaca`.

[2] J. B. Almeida, J. S. Pinto, and M. Vilaça. Token-passing Implementations of Recursion and Structured Types. In *Proceedings of the 7th International Workshop on Reduction Strategies in Rewriting and Programming (WRS'07)*, 2007. To appear in Elsevier ENTCS.

[3] J. B. Almeida, J. S. Pinto, and M. Vilaça. A Tool for Programming with Interaction Nets. In *Proceedings of the The Eighth International Workshop on Rule-Based Programming (RULE'07)*, 2007. To appear in Elsevier ENTCS.

[4] A. Asperti and S. Guerrini. *The Optimal Implementation of Functional Programming Languages*, volume 45 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1998.

[5] R. Bird and O. de Moor. *Algebra of Programming*. Prentice Hall, 1997.

[6] L. Dami and D. Vallet. Higher-order functional composition in visual form. Technical report, 1996.

[7] M. Fernández and I. Mackie. Operational equivalence for interaction nets. *Theoretical Computer Science*, 297(1–3):157–181, February 2003.

[8] J.-Y. Girard, Y. Lafont, and P. Taylor. *Proofs and Types*, volume 7 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 1989.

[9] G. Gonthier, M. Abadi, and J.-J. Lévy. The geometry of optimal lambda reduction. In *Proceedings of the 19th ACM Symposium on Principles of Programming Languages (POPL'92)*, pages 15–26. ACM Press, Jan. 1992.

[10] K. Hanna. Interactive Visual Functional Programming. In S. P. Jones, editor, *Proc. Intnl Conf. on Functional Programming*, pages 100–112. ACM, October 2002.

[11] J. Kelso. *A Visual Programming Environment for Functional Languages*. PhD thesis, Murdoch University, 2002.

[12] Y. Lafont. Interaction nets. In *Proceedings of the 17th ACM Symposium on Principles of Programming Languages (POPL'90)*, pages 95–108. ACM Press, Jan. 1990.

[13] I. Mackie. The geometry of interaction machine. In *Proceedings of the 22nd ACM Symposium on Principles of Programming Languages (POPL'95)*, pages 198–208. ACM Press, January 1995.

[14] I. Mackie. YALE: Yet another lambda evaluator based on interaction nets. In *Proceedings of the 3rd International Conference on Functional Programming (ICFP'98)*, pages 117–128. ACM Press, 1998.

[15] I. Mackie. Efficient λ-evaluation with interaction nets. In V. van Oostrom, editor, *Proceedings of the 15th International Conference on Rewriting Techniques and Applications (RTA'04)*, volume 3091 of *Lecture Notes in Computer Science*, pages 155–169. Springer-Verlag, June 2004.

[16] I. Mackie, J. S. Pinto, and M. Vilaça. Functional Programming and Program Transformation with Interaction Nets. In P. M. Hill, editor, *International Symposium on Logic-based Program Synthesis and Transformation (LOPSTR)*, 2005. Informal Proceedings.

[17] I. Mackie, J. S. Pinto, and M. Vilaça. Visual programming with recursion patterns in interaction nets. In K. Ehrig and H. Giese, editors, *Proceedings of the 6th International Workshop on Graph Transformation and Visual Modeling Techniques (GT-VMT'07)*, volume 6 of *Electronic Communications of the EASST*, 2007. ISSN 1863-2122.

[18] H. J. Reekie. *Realtime Signal Processing – Dataflow, Visual, and Functional Programming*. PhD thesis, University of Technology at Sydney, 1995.

[19] F.-R. Sinot. Call-by-name and call-by-value as token-passing interaction nets. In P. Urzyczyn, editor, *TLCA*, volume 3461 of *Lecture Notes in Computer Science*, pages 386–400. Springer, 2005.

[20] M. Vilaça. Inblobs webpage. http://haskell.di.uminho.pt/jmvilaca/INblobs/.

A Example Evaluation

The following represents some snapshots of the evaluation of the program

$$(\lambda xy.\mathsf{iternat}(\lambda r.\mathsf{suc}(r), y, x))\,(\mathsf{suc}(0))\,(\mathsf{suc}(0))$$